Bounding Functions and Rigid Graphs

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Abstract

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**keywords** graphs, extremal graph theory

**AMS subject classification** 05C99
BOUNDING FUNCTIONS AND RIGID GRAPHS*

MICHAEL O. ALBERTSON† AND RUTH HAAS†

Abstract. A function $f$ bounds graphs from above if there exists an infinite family of graphs $\mathcal{G}$, such that if $G \in \mathcal{G}$ then $f(|V_G|) = |E_G|$ and for all nonempty subgraphs $H$ of $G$ we have that $f(|V_H|) \geq |E_H|$. This paper considers the question: Which functions bound graphs?

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1. Introduction. Let $V_G$ ($E_G$) denote the vertex (edge) set of a simple graph $G$. A function $f$ is said to bound graphs from above if there exists an infinite family of graphs $\mathcal{G}$, such that if $G \in \mathcal{G}$, then $f(|V_G|) = |E_G|$ and for all nonempty subgraphs $H$ of $G$ we have that $f(|V_H|) \geq |E_H|$. If we insist on inequality for all proper subgraphs, we say that $f$ strictly bounds.

In this context we say that a family of graphs is bounded (strictly bounded). We will also speak of a single graph being bounded when its membership in a particular family is implicit. For example, the function $f(n) = n$ strictly bounds the cycles $C_n$, since any proper subgraph of $C_n$ with $k$ vertices contains fewer than $k$ edges. The function $f(n) = \binom{n}{2}$ bounds (but not strictly) since the graphs in $\mathcal{G}$ must be complete.

Some graph properties are characterized by bounding functions. For instance, a graph is a Hamilton cycle if and only if it is strictly bounded by $f(n) = n$. Not surprisingly, a tree is characterized by $f(n) = n - 1$. More generally, we have the following result.

THEOREM 1.1. A graph is the edge disjoint union of $k$ spanning trees if and only if it is bounded by the function $f(n) = k(n - 1)$.

Proof. Suppose $G$ can be decomposed into $k$ disjoint spanning trees. Since any subgraph, say $H$, of $G$ will contain a forest from each of these $k$ trees, $|E_H| \leq k(|V_H| - 1)$.

On the other hand, suppose that the $G$ is not decomposable into $k$ disjoint spanning trees. If $|E_G| \neq k|V_G| - k$ then clearly $G$ is not bounded by $k(n - 1)$. So we must only consider the case where $|E_G| = k|V_G| - k$. We invoke a result of Tutte.

THEOREM 1.2 (see [3]). A graph is the union of $k$ disjoint spanning trees if and only if for every partition of the vertices of $G$ into $m$ nonempty parts, there are at least $k(m - 1)$ edges between parts.

Hence, there exists a partition of the vertices of $G$ into $m$ parts such that the number of edges between parts is strictly less than $km - k$. Thus the total number of edges from within the parts is at least $k(m - |V_G|) + 1$. By the pigeon hole principle at least one part, say $H$, must have more than $k|V_H| - k$ edges.

Originally, we were intrigued by the role of bounding functions in rigidity theory. Suppose we construct the graph $G$ in the plane using inflexible bars for the edges and rotateable joints for the vertices. This immersion is rigid if the only motions of this structure are trivial (rotations and translations). The graph $G$ is said to be rigid in the plane if some immersion of it is rigid. A graph is minimally rigid if the removal of any edge makes it nonrigid. See [1], [2], and [4] for further information on rigidity.

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There is a combinatorial characterization of planar rigidity which in our language is stated as follows.

**Theorem 1.3** (Laman, see [2, p. 243]). A graph is minimally rigid in the plane if it is bounded above by \( f(n) = 2n - 3 \) for subgraphs on two or more vertices.

In higher dimensions a minimally rigid graph must have a bounded number of edges, but this condition is not sufficient. Specifically, if a graph is minimally rigid in 3 space then it is bounded above by \( f(n) = 3n - 6 \).

There are other properties of graphs for which belonging to a family bounded by a function is only necessary and not sufficient. For example, maximal planar graphs must be bounded by \( f(n) = 3n - 6 \). Graphs that are decomposable into \( k \) disjoint Hamiltonian cycles must be strictly bounded by \( f(n) = kn \), but again this condition is not sufficient. For instance, there are graphs on 6 vertices with 12 edges that are strictly bounded by \( f(n) = 2n \) but will not have two disjoint Hamiltonian cycles.

It would be interesting to determine which graph properties are characterized by, or just imply, a bounding function. In this paper we address a more fundamental issue, namely which functions bound graphs. Section 2 constructs families of graphs bounded by every reasonable linear function. Section 3 provides constructions for many quadratic functions. Section 4 considers functions between linear and quadratic with only modest success. Section 5 provides an extended definition of bounding and examines the interrelationships between the definitions.

### 2. Linear functions

In this section we address the question of which linear functions bound graphs. Specifically, we obtain the following two theorems.

**Theorem 2.1.** Given rational \( c \geq 1 \) and integer \( d \geq 0 \), the function \( f(n) = cn + d \) strictly bounds graphs from above.

**Theorem 2.2.** Given \( c \) a positive multiple of \( \frac{1}{2} \) and integer \( d \geq -c \), the function \( f(n) = cn + d \) bounds graphs from above.

The condition that \( d \geq c \) is in some sense the best possible. If \( d < -c \) then \( f(1) < 0 \), and the inequality would fail for subgraphs of one vertex.

Our constructions depend on combining families of graphs while preserving the fact that they are bounded. The simplest such combination is an edge union. If \( G \) and \( G^* \) are edge disjoint graphs on the vertex set \( V \) with edge sets \( E \) and \( E^* \), then we define the graph \( G + G^* \) to be the graph whose edge set is \( E \cup E^* \). If \( G \) and \( G^* \) are families of graphs such that whenever \( G \in G \) and \( G^* \in G^* \) have the same number of vertices, there are vertex labelings that make \( G \) and \( G^* \) edge disjoint; then we define \( G + G^* \) to be the family of graphs whose individuals are \( G + G^* \).

**Theorem 2.3.** If \( G \) and \( G^* \) are families of pairwise edge disjoint graphs bounded above by \( f \) and \( f^* \), then \( G + G^* \) is bounded above by \( f + f^* \). Moreover, if \( f \) or \( f^* \) bounds strictly then so does \( f + f^* \).

**Proof.** Let \( H \) be a subgraph of \( G + G^* \) on some vertex set \( V_H \subset V_G \). Each edge of \( E_H \) comes either from \( E_G \) or \( E_G^* \), but not both. Thus \( |E_H| = |E_{H_G}^*| + |E_{H_G^*}| \leq f(|V_H|) + f^* (|V_{H_G^*}|) = (f + f^*)(|V_H|) \). If at least one of \( f \) or \( f^* \) bounds strictly then the inequality is strict. \( \square \)

From this theorem we can immediately construct families of graphs with \( n \) vertices (for \( n \) sufficiently large) that are strictly bounded above by \( f(n) = cn \) for any integer constant \( c \). These are graphs whose edge sets are decomposable into \( c \) Hamiltonian cycles. A more general result can also be obtained using perfect matchings.

**Corollary 2.4.** Any family of graphs, each of whose members consists of \( c \) perfect matchings (and is thus \( c \)-regular), is bounded above by \( f(n) = \frac{3n}{2} \).

Let \( G \) be a family of graphs (strictly) bounded by \( f(n) \). If \( G \in G \) has \( n \) vertices and \( f(n) \leq \binom{n}{2} - d \), then we can construct a graph \( G^* \) with \( n \) vertices and \( d \) edges
that is edge disjoint from \( G \). If there is an infinite number of \( G \)'s such that this can be done, then by Theorem 2.3 \( f(n) + d \) (strictly) bounds the family \( \mathcal{G} + \mathcal{G}^* \). For example, if we take \( f(n) = cn \) for positive integer \( c \), then \( cn + d \) bounds for all positive integers \( c \) and \( d \).

To prove Theorem 2.1 it suffices to show that for rational \( c \geq 1 \), \( f(n) = cn \) strictly bounds. We construct a family of strictly \( f \) bounded graphs where \( f(n) = (1 + \frac{p}{q})n \) where \( p \) and \( q \) are integers and \( p < \frac{q}{2} \). These graphs will have vertices of degree 2 and 3 only. Since every graph with minimum degree at least \( \frac{2}{3} \) has a Hamiltonian cycle, the complement of such a graph on \( n \) vertices will contain at least \( \frac{2}{3} - 2 \) disjoint Hamiltonian cycles.

![Fig. 1. The ladder for \( p = 3, q = 11, \) and \( t = 1 \).](image)

Fix such a \( p \) and \( q \). For each integer \( t \) we construct \( G_t \), a graph with \( n = 2tp \) vertices and \( (1 + \frac{p}{q})n \) edges. The construction begins with the Cartesian product of an edge and the cycle \( C_{2tp} \). Alternatively one can imagine two disjoint copies of \( C_{2tp} \) where corresponding vertices in the two cycles are adjacent. This graph is called a ladder and the edges between the two cycles are called rungs. The remaining \( t(2q - 4p) \) vertices will be partitioned into \( 2tp \) parts, each part containing either \( \lfloor (q - 2p)/p \rfloor \) or \( \lceil (q - 2p)/p \rceil \) vertices. Each rung is subdivided by the vertices of one part. The resulting graph \( G_t \) contains \( 2tp \) vertices, \( t(2q - 4p) \) of degree 2, and \( 4tp \) of degree 3. Thus, it contains \( (1 + \frac{p}{q})2tp \) edges. Figure 1 illustrates this construction with \( p = 3, q = 11, \) and \( t = 1 \). To see that \( G_t \) is strictly bounded we look at the average degree of its subgraphs. If a graph has average degree greater than 2 then its average degree can be increased by deleting all vertices of degree 1. The reader may verify that any proper subgraph of \( G_t \) that contains no vertices of degree 1 must have lower average degree. Consequently, \( G_t \) is strictly bounded.

To prove Theorem 2.2 we use Theorem 2.3 to construct a family of graphs bounded by \( cn - d \) for positive integer \( d \leq c \) and \( c \) a positive multiple of \( \frac{1}{2} \). Graphs in this family are those that are decomposable into \( d \) Hamiltonian paths and \( 2(c - d) \) perfect matchings.

3. Quadratic functions. Definition 3.1. Given a graph \( G \) on \( n \) vertices we compose \( k \) copies of \( G \) as follows. Make \( k \) vertex disjoint copies of \( G \) and from each vertex in a particular copy of \( G \) put an edge to all vertices in all other copies of \( G \). This gives a graph with a total of \( m = kn \) vertices. This new graph will be denoted by the symbol \( K_k(G) \).

Theorem 3.2. Suppose \( \mathcal{G} \) is a family of graphs bounded by the function \( f(n) \) where \( f'' \leq 1 \). Then the graph \( K_k(G) \) is bounded by the function \( F(m) := kf\left(\frac{m}{k}\right) + \left(\frac{k-1}{k}\right) \frac{m^2}{2} \). Further, if \( f(n) \) strictly bounds then so does \( F(m) \).
Proof. It is straightforward to check that \( K_k(G) \) contains \( F(m) \) edges. We show the case where \( f(n) \) and consequently \( F(m) \) bounds strictly. To show that the family is strictly bounded we look at all subgraphs on \( V \) vertices \((V < m)\). Suppose the subgraph consists of \( v_i \) vertices from the \( i \)th copy of \( G \), \( i = 1, \ldots, k \) and \( v_i > 0 \) for all \( i \). The edges of the subgraph come from within the copies of \( G \) and between the copies of \( G \). The number of edges that come from between is given by \( \frac{1}{2} \Sigma v_i (V - v_i) \), and the number of edges that come from within the copies of \( G \) is less than \( \Sigma f(v_i) \). Combining these we get that the total number of edges in the subgraph is less than

\[
\left( \frac{1}{2} \right) \Sigma v_i (V - v_i) + \Sigma f(v_i).
\]

We will show that this function achieves its maximum value when all \( v_i \) are equal. The value at the maximum is then

\[
\left( \frac{1}{2} \right) k \frac{V}{k} \frac{(k-1)W}{k} + k f \left( \frac{V}{k} \right) = F(V).
\]

Consider two copies of \( G \) in \( K_k(G) \), say \( G_i \) and \( G_j \). The number of edges from these copies to all the others is equal regardless of how the vertices are divided between \( G_i \) and \( G_j \). Suppose \( G_i \) has \( W - t \) vertices and \( G_j \) has \( W + t \) vertices. \((W \text{ need not be an integer.})\) The number of edges in \( G_i \) and \( G_j \) and between these two parts is then less than

\[
g(t) = (W - t)(W + t) + f(W - t) + f(W + t).
\]

We use the first degree Taylor polynomial approximation of \( f \) near \( W \) to get that

\[
g(t) = W^2 - t^2 + 2f(W) + f'(W)(-t + t) + \frac{(-t)^2}{2}f''(c_1) + \frac{t^2}{2}f''(c_2)
\]

for some \( W - t < c_1 < W \) and \( W < c_2 < W + t \). By assumption, \( f''(c) \leq 1 \) for any value of \( c \). Thus, \( g(t) \leq W^2 + 2f(W) \).

Hence, the number of edges that can come from within and between just two parts is bounded by the case that these two parts have an equal number of vertices. Since this argument can be made for any pair of copies of \( G \) the maximum number of edges occurs when all the parts have an equal number of vertices. \( \Box \)

4. \( \frac{3}{2} \log_2 n + n \) bounds. Let \( Q_r \) denote the standard \( r \)-cube. \( Q_r \) has \( 2^r \) vertices and \( 2^{r-1} \) edges. So a cube with \( n \) vertices has \( \frac{n}{2} \log_2 n \) edges.

THEOREM 4.1. The cubes form a family that is bounded by \( f(n) = \frac{3}{2} \log_2 n \).

Proof. The proof will be by induction on the dimension of the cube. The base case is elementary. Suppose \( H \) is a subgraph of \( Q_r \) with \( m \) vertices. Let \( H_1 \) be the restriction of \( H \) to those vertices whose first coordinate in \( Q_r \) is 1, while \( H_2 \) is the restriction of \( H \) to those vertices whose first coordinate in \( Q_r \) is 0. Both \( H_1 \) and \( H_2 \) are subgraphs of \( Q_{r-1} \). Let \( m \) denote \( |V_{H_1}| \). Then \( |V_{H_2}| = m - m_1 \). We may assume that \( m_1 \leq m - m_1 \). The edges in \( H \) are either in \( H_1 \), in \( H_2 \), or between these two. Since \( H_1 \) and \( H_2 \) are bounded by the inductive hypothesis, we obtain

\[
|E_H| \leq \frac{m_1}{2} \log_2 m_1 + \frac{m - m_1}{2} \log_2 (m - m_1) + m_1.
\]

It is straightforward to see that the second derivative of the right-hand side of the preceding equation is positive. Thus a maximum value must occur at one of the endpoints. From this it is immediate that \( |E_H| \leq \frac{3}{2} \log_2 m \). \( \Box \)

By Theorem 2.3, for large enough \( n \) we can add a Hamiltonian cycle to \( Q_r \). We then obtain the following.

COROLLARY 4.2. \( f(n) = \frac{3}{2} \log_2 n + n \) bounds strictly.
5. Extensions. In our original example of rigidity, the bounding condition only needed to hold for subgraphs on two or more vertices. This suggests that we should use a broader definition for functions that bound from above.

Extended definition. A function $f$ bounds graphs from above if there exists an infinite family of graphs $\mathcal{G}$ and an integer $n_0 > 0$ such that if $G \in \mathcal{G}$ then $f(|V(G)|) = |E(G)|$, and for all subgraphs $H$ of $G$ with at least $n_0$ vertices we have that $f(|V(H)|) \geq |E(H)|$. As before, we say $f$ strictly bounds if the inequality is strict for all proper subgraphs with at least $n_0 > 0$ vertices.

Most of our results still hold under this extended definition. We will get a larger class of functions that bound and that strictly bound this way. For example, the function $f(n) = an - (2n - 1)$, with $a \geq 3$ an integer, strictly bounds graphs from above as witnessed by the family $C_{n-2} \vee K_2$.

Let $O$ be the set of functions that strictly bound under the original definition, $E$ the set of functions that bound under the original definition, and $SE$ the set of functions that strictly bound under the extended definition. Figure 2 shows the relationships between these.

We show that the sets are arranged as indicated by the figure. It is clear that any function that bounds strictly also bounds and that any function that bounds under the original definition will bound under the extended definition. The function $f(n) = n - 1$ is in $O$ and not in $SE$, while $2n - 4$ is in $SE$ but not in $O$. Functions of the form $cn - c$ for $c \geq 2$ are in both $SE$ (for subgraphs on two or more vertices) and $O$, and not in $S$. We have no examples to show that $SE \cup O \neq E$, but we suspect this to be so. The functions $n - d$ for $d \geq 2$ are examples of increasing functions for which there are graphs with $n$ vertices and $n - d$ edges, but yet these are not bounding functions.

REFERENCES